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# CONSUMPTION AND INVESTMENT STRATEGIES WITH HYPERBOLIC DISCOUNTING AND LABOR INCOME

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Abstract. We investigate the optimal consumption and investment decision problem of an agent whose time preference is time-inconsistent. Specifically, for a time-separable utility function, the agent's subjective discount factor is supposed to be changed randomly in the future. We provide closed-form solutions in the presence of income process. The method can be extended into the case with a stochastic income process.

## 1. Introduction

We extend the classical Merton's framework into the model with stochastic discounting rate, where there is a jump on the subjective discount rate and that jump is supposed to follow a Poisson distribution with a constant intensity. We call this kind of discounting as a hyperbolic-discounting.

The standard hyperbolic-discounting model is developed in Phelps and Pollak (1968) and Laibson [2] in discrete-time setting. The continuous time models which extends Merton's model with hyperbolic discounting are considered in Marín-Solano and Navas [3], Harris and Laibson [1], Palacios-Huerta and Pérez-Kakabadse [6], and Zhou et al. [8]. However, none of the studies incorporates a labor wage. The difficulty to get an explicit solutions might be the main reason. In this paper, we consider the hyperbolic discounting model with income process and

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provide the closed-form solutions. Both constant and stochastic income processes are investigated.

The paper is organized as follows. Section 2 introduces the model with hyperbolic discounting. Section 3 derives the Hamilton-Jacobi-Bellman (HJB) equation of the value function. The closed-form solutions of the model with constant and stochastic income processes are provided in Section 4 and 5 respectively. Section 6 provides the concluding remarks.

## 2. The Model

### 2.1. Preference

We assume that an agent's preference has a CRRA utility which is defined by

$$u(c_t) = \frac{1}{1-\gamma} c_t^{1-\gamma}, \qquad \gamma > 0, \gamma \neq 1.$$

In addition, time preference is supposed to be inconsistent. In words, the current selves' discount rate is different from that of future selves. We introduce the quasi-hyperbolic discounting such as

$$D(t,s) = \begin{cases} e^{-\delta(s-t)}, & s \in (t, \tau_t) \\ \beta e^{-\delta(s-t)}, & s \in (\tau_t, \infty), \end{cases}$$

where $\tau_t$ is the random time to switch the discounting rate. It is supposed that the time preference change is an independent risk source and follows Poisson process with a constant intensity $\lambda > 0$. Thus, the probability of a time preference change is defined by $\mathbb{P}(\tau_t > s) = 1 - e^{-\lambda(s-t)}$, for $s > t$.

### 2.2. Financial Market

In continuous time financial market, there exist two assets, which are a risky asset, $S_t$, and risk-free asset, $S_t^0$. The risk-free interest rate is given by a positive constant $r$ so the dynamics of risk-free asset is given by

$$dS_t^0 = r S_t^0 dt.$$

The risky asset follows a geometric Brownian motion with drift $\mu_s$ and volatility $\sigma_s$, i.e.,

$$dS_t = \mu_s S_t dt + \sigma_s S_t dB_t,$$

where $B_t$ is the standard Brownian motion under the probability space $(\Omega, \mathbb{P}, \mathcal{F})$.

The agent is supposed to receive an income from labor. The income process is also assumed to follow a geometric Brownian motion

$$dI_t = \mu_I I_t dt + \sigma_I I_t dB_t, \qquad I_0 = \bar{I}, \tag{2.1}$$

where $\mu_I$ and $\sigma_I$ are mean the growth rate and volatility respectively. Note that for tractability, we assume that the income and market risks are perfectly correlated.

Let us denote the optimal consumption rate and portfolio amount by $c_t$ and $\pi_t$ respectively. We assume that the consumption rate process is $\mathcal{F}_t$-progressively measurable and integrable almost surely (a.s.) and the portfolio process is $\mathcal{F}_t$-measurable and square integrable a.s., i.e.,

$$\int_t^\infty c_s ds < \infty, \text{ a.s.}, \qquad \int_t^\infty \pi_s^2 ds < \infty, \text{ a.s.}.$$

Then, the wealth dynamics evolves

$$dX_t = (rX_t + \pi_t(\mu_s - r) - c_t + I_t)dt + \sigma_s \pi_t dB_t. \tag{2.2}$$

### 2.3. The Problem

The value function at time $t$ is defined as follows.

$$V(X_t) = \sup_{c_t, \pi_t} \mathbb{E}_t \left[ \int_t^{t+\tau_t} e^{-\delta(s-t)} \frac{c_s^{1-\gamma}}{1-\gamma} ds + \beta e^{-\delta \tau_t} u(X_{t+\tau_t}) \right], \tag{2.3}$$

where $u(X_t)$ is the value function of future selves defined as

$$u(X_t) = \mathbb{E}_t \left[ \int_t^\infty e^{-\delta(s-t)} \frac{c_s^{*1-\gamma}}{1-\gamma} ds \right].$$

The consumption $c_t^*$ represents the optimal level chosen by future selves.

In Zho et al. [8], the Hamilton-Jacobi-Bellman (HJB) equation of continuous time problem is derived as the convergence result of discrete time approach. In the presence of income process, we can also apply the similar procedure with a slight modification even when the income process is stochastic as in (2.2). Instead of the convergence results, we apply more intuitive method developed in Palacios-Huerta and Pérez-Kakabadse [6].

The recursive representation of the value function $V(x)$ is rewritten as

$$V(x) = \frac{c(x)^{1-\gamma}}{1-\gamma} dt + e^{-\lambda dt} \mathbb{E}\left[ e^{-\delta dt} V(x + dx) \right] + (1 - e^{-\lambda dt}) \mathbb{E}\left[ e^{-\delta dt} \beta u(x + dx) \right].$$

If we divide by $dt$ on both sides, we can obtain the following equation

$$\delta V(x) = \frac{c(x)^{1-\gamma}}{1-\gamma} + \frac{\mathbb{E}\left[dV(x)\right]}{dt} + \lambda(\beta u(x) - V(x)). \tag{2.4}$$

From Itô's formula, the second term in the right-hand side is unfolded by

$$\begin{aligned}\mathbb{E}[dV(x)] &= \mathbb{E}[V'(x)(rx + \pi_t(\mu_s - r) - c_t + I_t)dt + V'(x)(\sigma_s \pi_t dB_t)] \\ &= V'(x)(rx + \pi_t(\mu_s - r) - c_t + I_t)dt \end{aligned} \tag{2.5}$$

Therefore, if we substitute (2.5) into (2.4) we can obtain the HJB equation for $V(x)$. The following lemma summarizes the result in Palacios-Huerta and Pérez-Kakabadse [6].

Lemma 2.1. *The value function $V(x)$ in (2.3) should satisfy the following HJB equation*

(2.6)

$$\delta V(x) + H(x) = \sup_{c_t, \pi_t} \left\{ \frac{c_t^{1-\gamma}}{1-\gamma} + (rx + \pi_t(\mu_s - r) - c_t + I_t)V'(x) + \frac{1}{2}\sigma_s^2 \pi_t^2 V''(x), \right\}$$

*where*

$$H(x) = \lambda(1-\beta)\mathbb{E}\left[\int_0^\infty e^{-(\lambda+\delta)t} \frac{c_t^{*\,1-\gamma}}{1-\gamma} ds\right], \tag{2.7}$$

*and $c_t^*$ is the optimal consumption rate which is the same as that of the right-hand side of HJB equation (2.6).*

*In addition, in the absent of labor income, the value function, the optimal consumption rate and investment are obtained from*

$$V(x) = \frac{1}{K^\gamma(1-\gamma)} x^{1-\gamma}, \quad c_t^* = KX_t, \quad \pi_t^* = \frac{\theta}{\gamma\sigma_s} X_t,$$

*where $\theta = (\mu_s - r)/\sigma_s$ and the constant $K$ satisfies the following algebraic equation*

$$\gamma K = \delta - (1-\gamma)\left(r + \frac{\theta^2}{2\gamma}\right) + \frac{\gamma(1-\beta)K}{\delta + \lambda - (1-\gamma)\left(r - K + \frac{\theta^2}{2\gamma}\right)}.$$

*Proof.* We refer Palacios-Huerta and Pérez-Kakabadse [6] to obtain $H(x)$ and the solutions to the case without labor income. □

For the well-defined optimal consumption and investment, we impose the following assumptions throughout the paper.

Assumption 2.2. $\gamma > 1 - \beta$.

ASSUMPTION 2.3. $r + \frac{\beta - r}{\gamma} + \frac{(\gamma-1)}{2\gamma^2}\theta^2 > 0.$

ASSUMPTION 2.4. $\lim_{s\to\infty} \mathbb{E}_t\left[\exp(-\delta t)u(X_s)\right] = 0.$

Note that the constant $K$ is determined implicitly and can be rewritten as

$$K = \frac{1}{\gamma}\left(\delta + \frac{\lambda(1-\beta)K}{\lambda + \delta + (1-\gamma)(K - \bar{\mu} + \frac{1}{2}\gamma\bar{\sigma}^2)} - (1-\gamma)(\bar{\mu} - \frac{1}{2}\gamma\bar{\sigma}^2)\right),$$

where $\bar{\mu}$ and $\bar{\sigma}$ are mean and variance of the return on the optimal portfolio, which are defined by

$$\bar{\mu} = \pi^*\mu + (1-\pi^*)r, \quad \bar{\sigma} = \pi^*\sigma_s.$$

The first two terms represent the effective discount rate and it is greater than current selves' discount rate, $\delta$ since the second term in parenthesis is positive due to Assumption 2.2 and 2.3. Notice that when $\lambda = 0$ or $\beta = 1$, the constant $K$ is reduced to the Merton's constant $M$, which is defined by

$$M = \frac{1}{\gamma}\left(\delta - (1-\gamma)(\bar{\mu} - \frac{1}{2}\gamma\bar{\sigma}^2)\right).$$

## 3. The Solution: Constant Income

In this section, we extend the model with income stream. The only difference from the previous section is the existence of income stream. Let us denote the constant income stream of an agent by $\epsilon$. Then the wealth dynamics is unfolded by

$$dX_t = (rX_t + \pi_t(\mu - r) - c_t + \epsilon)dt + \sigma\pi_t dB_t.$$

If we denote the value function in the presence constant income stream by $V_\epsilon(x)$, we have the equation derived in (2.4) due to the fact that same recursive representation of the value function, $V(x)$, can be applied even with $\epsilon$. In addition, the expected value of $dV_\epsilon(x)$ is given by

$$\mathbb{E}[dV_\epsilon(x)] = V_\epsilon'(x)(rx + \pi_t(\mu_s - r) - c_t + \epsilon),$$

and $V_\epsilon(x)$ should satisfy the following HJB equation

(3.1)

$$\delta V_\epsilon(x) + H_\epsilon(x) = \sup_{c_t,\pi_t}\left\{\frac{c_t^{1-\gamma}}{1-\gamma} + (rx + \pi_t(\mu_s - r) - c_t + \epsilon)V_\epsilon'(x) + \frac{1}{2}\sigma_s^2\pi_t^2 V_\epsilon''(x)\right\},$$

where $H_\epsilon(x)$ is same as the expectation in (2.7).

Now, to obtain the closed-form solution we conjecture the value function by

$$V_\epsilon(x) = \frac{1}{\tilde{K}^\gamma(1-\gamma)}\left(x + \frac{\epsilon}{r}\right)^{1-\gamma}.$$

Then, the optimal consumption rate and portfolio to the HJB equation (3.1) are determined by

$$\begin{aligned}\tilde{c}_t^* &= \left(V_\epsilon'(x)\right)^{-\frac{1}{\gamma}} = \tilde{K}\left(X_t + \frac{\epsilon}{r}\right)\\ \tilde{\pi}_t^* &= -\frac{\theta V_\epsilon'(x)}{\sigma V_\epsilon''(x)} = \frac{\theta}{\gamma\sigma}\left(X_t + \frac{\epsilon}{r}\right).\end{aligned}$$

By substituting the optimal controls into (3.1), we obtain

$$\frac{\delta\tilde{K}}{(1-\gamma)} + \frac{\tilde{K}^\gamma H_\epsilon(x)}{\left(x+\frac{\epsilon}{r}\right)^{1-\gamma}} = \frac{\tilde{K}}{1-\gamma} + \left(r + \frac{\theta^2}{\gamma} - \tilde{K}\right) - \frac{\theta^2}{2\gamma},$$

where $H_\epsilon(X_t), s > t$, is given by

$$H_\epsilon(X_t) = \lambda(1-\beta)\mathbb{E}_t\left[\int_t^\infty e^{-(\lambda+\delta)(s-t)}\frac{(\tilde{K}(X_s+\epsilon/r))^{1-\gamma}}{1-\gamma}ds\right]. \tag{3.2}$$

In the following lemma, we provide the value of $H_\epsilon(x)$.

LEMMA 3.1. *The function $H_\epsilon(x)$ in (3.2) has its explicit form as*

$$H_\epsilon(x) = \frac{\lambda(1-\beta)\tilde{K}^{1-\gamma}}{\lambda+\delta-(1-\gamma)\left(r-\tilde{K}+\frac{\theta^2}{2\gamma}\right)}\left(X_t+\frac{\epsilon}{r}\right)^{1-\gamma}.$$

*Proof.* The function $H_\epsilon(x)$ is rewritten as

$$\begin{aligned}H_\epsilon(X_t) &= \lambda(1-\beta)\mathbb{E}\left[\int_t^\infty e^{-(\lambda+\delta)(s-t)}\frac{(\tilde{K}(X_s+\epsilon/r))^{1-\gamma}}{1-\gamma}ds\right]\\ &= \lambda(1-\beta)\frac{\tilde{K}^{1-\gamma}}{1-\gamma}\int_t^\infty e^{-(\lambda+\delta)(s-t)}\mathbb{E}_t\left[\left(X_t+\frac{\epsilon}{r}\right)^{1-\gamma}\right]ds\end{aligned}$$

It is necessary to derive the explicit form of $X_t + \epsilon/r$. If we substitute the optimal controls into wealth dynamics, we have

$$d\left(X_t+\frac{\epsilon}{r}\right) = \left(r+\frac{\theta^2}{\gamma}-\tilde{K}\right)\left(X_t+\frac{\epsilon}{r}\right)dt + \frac{\theta}{\gamma}\left(X_t+\frac{\epsilon}{r}\right)dB_t.$$

and it implies that

$$X_t + \frac{\epsilon}{r} = \left(x+\frac{\epsilon}{r}\right)\cdot\exp\left\{\left(\gamma+\frac{\theta^2}{\gamma}-\tilde{K}-\frac{\theta^2}{2\gamma^2}\right)t+\frac{\theta}{\gamma}B_t\right\}$$

Thus, $\mathbb{E}_t\left[\left(X_s+\frac{\epsilon}{r}\right)^{1-\gamma}\right]$ for $s>t$, can be calculated by

$$\mathbb{E}_t\left[\left(X_s+\frac{\epsilon}{r}\right)^{1-\gamma}\right]=\left(X_t+\frac{\epsilon}{r}\right)^{1-\gamma}\exp\left\{\left((1-\gamma)(r-\tilde{K}+\frac{(1-\gamma)\theta^2}{2\gamma})\right)(s-t)\right\}.$$

We have the result. □

PROPOSITION 3.2. *When an agent who has a hyperbolic discounting rate receives a stochastic income stream, the optimal consumption rate and investment are obtained from*

$$\tilde{c}_t^*=\tilde{K}\left(X_t+\frac{\epsilon}{r}\right),$$

$$\tilde{\pi}_t^*=\frac{\theta}{\gamma\sigma}\left(X_t+\frac{\epsilon}{r}\right),$$

*where the constant $\tilde{K}$ satisfies the following algebraic equation*

$$\gamma\tilde{K}=\delta-(1-\gamma)\left(r+\frac{\theta^2}{2\gamma}\right)+\frac{\gamma(1-\beta)\tilde{K}}{\delta+\lambda-(1-\gamma)\left(r-\tilde{K}+\frac{\theta^2}{2\gamma}\right)}.$$

Notice that the constant $\tilde{K}$ is exactly same with that of the problem without income stream, which implies that the existence of income stream does not have any impact on the marginal effect on consumption rate, which is described by $\tilde{K}$.

## 4. Stochastic Income

We consider the stochastic income stream in this section. The preference and financial markets are supposed to be identical to the case of no income. For the tractability, we also assume that the income process follows a geometric Brownian motion with the same uncertainty as risky asset. Specifically, the income process denoted by $I_t$ evolves

$$\frac{dI_t}{I_t}=\mu_I dt+\sigma_I dB_t,\qquad I_0=\bar{I}.$$

Let us denote the value function with stochastic income by $V^I(x,I)$. Then it should satisfy the following HJB equation

$$\begin{aligned}\delta V^I(x,I)+H^I(x,I)=\sup_{c,\pi}\Bigg[&\frac{c^{1-\gamma}}{1-\gamma}+(rx+\pi(\mu_s-r)-c+I)V_x^I(x,I)+\mu_I I V_i^I(x,I)\\&+\frac{1}{2}V_{xx}^I(x,I)\sigma_s^2\pi^2+\frac{1}{2}V_{ii}^I(x,I)\sigma_I^2I^2+V_{xi}^I(x,I)\sigma_s\sigma_I\pi I\Bigg],\end{aligned}\tag{4.1}$$

where

$$H^I(x, I) = \lambda(1-\beta)\mathbb{E}\left[\int_0^\infty e^{-(\lambda+\delta)t} \frac{c_t^{I,*\,1-\gamma}}{1-\gamma} dt\right], \tag{4.2}$$

and $c_t^{I,*}$ is the optimal consumption rate which satisfies the RHS of HJB equation (4.1).

In a similar manner to the problem with a constant income stream, we conjecture the value function by

$$V^I(X_t, I_t) = \frac{1}{K_I^\gamma(1-\gamma)}\left(X_t + \frac{I_t}{r_I}\right)^{1-\gamma},$$

where $r_I = r - \mu_I + \theta\sigma_I$. Then, the optimal consumption rate and portfolio are rewritten as

$$c_t^{I,*} = V_x^I(X_t, I_t)^{-\frac{1}{\gamma}} = K_I\left(X_t + \frac{I_t}{r_I}\right), \tag{4.3}$$

$$\pi_t^{I,*} = -\frac{\theta V_x^I(X_t, I_t)}{\sigma V_{xx}^I(X_t, I_t)} - \frac{\sigma_I I_t V_{ix}^I(X_t, I_t)}{\sigma_s V_{xx}^I(X_t, I_t)} = \frac{\theta}{\gamma\sigma}\left(X_t + \frac{I_t}{r_I}\right) - \frac{\sigma_I}{r_I\sigma_s} I_t. \tag{4.4}$$

If we substitute the (4.3) and (4.4) into the HJB equation, it is reduced to

$$\delta V^I(x, I) + H^I(x, I) = \frac{1}{K_I^\gamma}\left(x + \frac{\bar{I}}{r_I}\right)\left(\frac{K_I}{1-\gamma} + \frac{\theta^2}{\gamma} - K_I + r - \frac{\theta^2}{\gamma}\right).$$

We provide the value $H^I(x, I)$ in the following lemma.

Lemma 4.1. *The function $H^I(x, \bar{I})$ in has a closed-from solution as*

$$H^I(X_t, I_t) = \frac{\lambda(1-\beta)K_I^{1-\gamma}}{\lambda + \delta - (1-\gamma)\left(r - K_I + \frac{\theta^2}{2\gamma}\right)}\left(X_t + \frac{I_t}{r_I}\right)^{1-\gamma}.$$

*Proof.* The function $H^I(X_t, I_t)$ is rewritten as

$$\begin{aligned} H^I(X_t, I_t) &= \lambda(1-\beta)\mathbb{E}\left[\int_t^\infty e^{-(\lambda+\delta)(s-t)} \frac{(K_I(X_t + I_t/r_I))^{1-\gamma}}{1-\gamma} ds\right] \\ &= \lambda(1-\beta)\frac{K_I^{1-\gamma}}{1-\gamma}\int_t^\infty e^{-(\lambda+\delta)(s-t)}\mathbb{E}_t\left[\left(X_t + \frac{I_t}{r_I}\right)^{1-\gamma}\right] ds \end{aligned}$$

It is necessary to derive the explicit form of $X_t + I_t/r_I$. If we substitute the optimal controls into wealth dynamics, we have

$$d\left(X_t + \frac{I_t}{r_I}\right) = \left(r + \frac{\theta^2}{\gamma} - K_I\right)\left(X_t + \frac{I_t}{r_I}\right)dt + \frac{\theta}{\gamma}\left(X_t + \frac{I_t}{r_I}\right)dB_t.$$

and it implies that

$$X_t + \frac{I_t}{r_I} = \left(X_t + \frac{I_t}{r_I}\right)\cdot\exp\left\{\left(\gamma + \frac{\theta^2}{\gamma} - K_I - \frac{\theta^2}{2\gamma^2}\right)t + \frac{\theta}{\gamma}B_t\right\}$$

Thus, $\mathbb{E}_t\left[\left(X_s + \frac{I_s}{r_I}\right)^{1-\gamma}\right]$ for $s > t$, can be calculated by

$$\mathbb{E}_t\left[\left(X_s + \frac{I_s}{r_I}\right)^{1-\gamma}\right] = \left(X_t + \frac{I_t}{r_I}\right)^{1-\gamma}\exp\left\{\left((1-\gamma)(r - K_I + \frac{(1-\gamma)\theta^2}{2\gamma})\right)(s-t)\right\}.$$

We have the result. □

We summarize our main results in the following proposition.

Proposition 4.2. *When an agent who has a hyperbolic discounting rate receives a constant income stream, the optimal consumption rate and investment are obtained from*

$$c_t^{I,*} = K_I\left(X_t + \frac{I_t}{r_I}\right),$$
$$\pi_t^{I,*} = \frac{\theta}{\gamma\sigma}\left(X_t + \frac{I_t}{r_I}\right) - \frac{\sigma_I}{r_I\sigma_s}I_t,$$

*where the constant $K_I$ satisfies the following algebraic equation*

$$\gamma K_I = \delta - (1-\gamma)\left(r + \frac{\theta^2}{2\gamma}\right) + \frac{\gamma(1-\beta)K_I}{\delta + \lambda - (1-\gamma)\left(r - K_I + \frac{\theta^2}{2\gamma}\right)}.$$

## 5. Conclusion

We obtain the closed-form solutions to the consumption and investment problem of an agent with time inconsistent preference. We extend the model into problems with labor income. Interestingly, it is verified that the marginal propensities to consume which are $K, \tilde{K}$, and $K_I$ are identical regardless of income process.

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